A State-Augmented Framework for Analyzing the Collatz Conjecture: Reduction to Eventual Descent

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Abstract

The Collatz conjecture asserts that repeated application of a simple arithmetic function to any positive integer eventually leads to 1. This paper introduces a state-augmented system $X = \mathbb{Z}^+_{\text{odd}} \times (\mathbb{Z}/4\mathbb{Z})$ to analyze the "shortcut" Collatz map, which operates on odd positive integers. By explicitly tracking an integer's residue modulo 4, we precisely determine the 2-adic valuation of $3s + 1$. We demonstrate that any sequence starting from an odd positive integer $s_0 > 1$ either directly reduces to a Collatz sequence on a smaller integer $m < s_0$ (if $s_0 \equiv 1 \pmod 4$), or, after a finite and bounded number of steps (at most $\text{val}_2(s_0 + 1) - 1$, if $s_0 \equiv 3 \pmod 4$), transitions to a state from which it reduces to a Collatz sequence on an integer m^*. The proof of the full Collatz conjecture then hinges on the property that this m^* (or a subsequent term in its own Collatz sequence) is eventually less than the original s_0, a property equivalent to the core "eventual descent" aspect of the Collatz conjecture. This framework thus provides a clear structural decomposition and reduction of the problem.

1 Introduction

The Collatz conjecture, also known as the $3n + 1$ problem, concerns the behavior of iterates of the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

The conjecture states that for any starting positive integer n, the sequence $n, f(n), f(f(n)), \ldots$ eventually reaches the integer 1.

It is often convenient to study a "shortcut" version of the Collatz map that jumps directly to the next odd integer.

Definition 1.1 (Shortcut Collatz Map T_C)**.** *For an odd positive integer $s \in \mathbb{Z}^+_{odd}$ (where $\mathbb{Z}^+_{odd} = \{n \in \mathbb{Z}^+ \mid n \text{ is odd}\}$), the shortcut Collatz map $T_C : \mathbb{Z}^+_{odd} \to \mathbb{Z}^+_{odd}$ is defined as*

$$T_C(s) = \frac{3s + 1}{2^{\text{val}_2(3s+1)}}$$

where $\text{val}_2(x)$ is the 2-adic valuation of x, i.e., the exponent of the highest power of 2 dividing x. The Collatz conjecture is equivalent to stating that for any $s_0 \in \mathbb{Z}^+_{odd}$, the sequence $s_0, s_1 = T_C(s_0), s_2 = T_C(s_1), \ldots$ eventually reaches 1. Note that $T_C(1) = 1$.

This paper introduces a state-augmented system to analyze the map T_C. By incorporating state information related to $s \pmod 4$, we can rigorously track the behavior of $\mathrm{val}_2(3s+1)$ and reduce the conjecture to a well-known descent property.

2 The State-Augmented System X

Definition 2.1 (State Space X). *Let $S = \mathbb{Z}/4\mathbb{Z} = \{[0],[1],[2],[3]\}$, representing residue classes modulo 4. The state space is defined as $X = \mathbb{Z}^+_{odd} \times S$. An element of X is an ordered pair (s,σ), where $s \in \mathbb{Z}^+_{odd}$ and $\sigma = [s \pmod 4] \in S$. Since s is odd, $\sigma \in \{[1],[3]\}$.*

The state component σ is crucial as it determines the primary behavior of $\mathrm{val}_2(3s+1)$.

Definition 2.2 (Valuation $k(s,\sigma)$ based on State). *For $(s,\sigma) \in X$:*

1. *If $\sigma = [1]$ (i.e., $s \equiv 1 \pmod 4$): Let $s = 4m+1$ for some integer $m \geq 0$. Then $3s+1 = 3(4m+1)+1 = 12m+4 = 4(3m+1)$. So $k(s,[1]) = \mathrm{val}_2(4(3m+1)) = 2 + \mathrm{val}_2(3m+1)$. Since $m \in \mathbb{Z}_{\geq 0}$, $3m+1$ can be even or odd, so $\mathrm{val}_2(3m+1) \geq 0$. Therefore, $k(s,[1]) \geq 2$.*

2. *If $\sigma = [3]$ (i.e., $s \equiv 3 \pmod 4$): Let $s = 4m+3$ for some integer $m \geq 0$. Then $3s+1 = 3(4m+3)+1 = 12m+9+1 = 12m+10 = 2(6m+5)$. Since $m \in \mathbb{Z}_{\geq 0}$, $6m$ is even, so $6m+5$ is an odd integer. Thus $\mathrm{val}_2(6m+5) = 0$. Therefore, $k(s,[3]) = \mathrm{val}_2(2(6m+5)) = 1$.*

Definition 2.3 (State-Augmented Operator T_X). *The operator $T_X : X \to X$ is defined for $(s,\sigma) \in X$ as:*

1. *Let $k = k(s,\sigma)$ be the valuation from Definition 2.2.*

2. *Calculate $s' = \frac{3s+1}{2^k}$. (Note s' is always an odd positive integer).*

3. *Determine the new state component $\sigma' = [s' \pmod 4]$.*

4. *Then $T_X(s,\sigma) = (s',\sigma')$.*

The state $(1,[1])$ is a fixed point of T_X. If $s = 1$, then $m = 0$ (since $1 = 4(0)+1$), so $\sigma = [1]$. $k(1,[1]) = 2 + \mathrm{val}_2(3(0)+1) = 2+0 = 2$. $s' = (3(1)+1)/2^2 = 4/4 = 1$. $\sigma' = [1 \pmod 4] = [1]$. Thus, $T_X(1,[1]) = (1,[1])$.

3 Convergence Analysis for Positive Integers

Theorem 3.1. *For any initial state (s_0,σ_0) where $s_0 \in \mathbb{Z}^+_{odd}$ and $\sigma_0 = [s_0 \pmod 4]$, the sequence $(s_i,\sigma_i) = T_X^i(s_0,\sigma_0)$ eventually reaches the fixed point $(1,[1])$ if and only if the Collatz conjecture is true. Specifically, the problem reduces to showing that an intermediate integer m^* generated during the process (or a subsequent term in its Collatz sequence) is eventually strictly less than s_0 for $s_0 > 1$.*

The proof relies on strong induction on s_0. The base case $s_0 = 1$ is established. Inductive Hypothesis (IH): Assume that for all odd positive integers j' such that $1 \leq j' < s_0$, the sequence generated by T_X starting from $(j',[j' \pmod 4])$ eventually reaches $(1,[1])$.

Lemma 3.2 (Descent Step for $\sigma = [1]$.]*Let(s, [1])beastatewiths* $\in \mathbb{Z}^+_{odd}$, $s > 1$, *so* $s \equiv 1 \pmod 4$. *Let* $T_X(s, [1]) = (s', \sigma')$. *Then* $s = 4m + 1$ *for some integer* $m = (s-1)/4 \geq 1$. *The next s-value is* $s' = T_C(m)$. *Critically,* $m < s$.

Proof. If $s \equiv 1 \pmod 4$ and $s > 1$, then $s = 4m + 1$ for some integer $m \geq 1$. From Definition 2.2, $k(s, [1]) = 2 + \mathrm{val}_2(3m + 1)$. Then $s' = \frac{3(4m+1)+1}{2^{2+\mathrm{val}_2(3m+1)}} = \frac{12m+4}{4 \cdot 2^{\mathrm{val}_2(3m+1)}} = \frac{4(3m+1)}{4 \cdot 2^{\mathrm{val}_2(3m+1)}} = \frac{3m+1}{2^{\mathrm{val}_2(3m+1)}} = T_C(m)$. Since $s = 4m + 1$ and $m \geq 1$ (as $s > 1$), it follows that $m < s$. □

Lemma 3.3 (Finite Transition from $\sigma = [3]$. *to*$\sigma = [1]$ *and Subsequent Descent Argument] Let* $(s_0, [3])$ *be a state with* $s_0 \in \mathbb{Z}^+_{odd}$, *so* $s_0 \equiv 3 \pmod 4$.

(a) *The sequence* $(s_i, \sigma_i) = T_X^i(s_0, [3])$ *will reach a state* $(s_{j_0}, [1])$ *for some* $j_0 \geq 1$. *This* j_0 *is the smallest such positive integer, and* $j_0 \leq \mathrm{val}_2(s_0 + 1) - 1$.

(b) *Let* $s^* = s_{j_0}$. *The next step according to Lemma 3.2 involves* $m^* = (s^* - 1)/4$. *For the Collatz conjecture to hold via this inductive path, the sequence starting from* m^* *must converge to 1, which is true by IH if* $m^* < s_0$. *If* $m^* \geq s_0$, *convergence relies on the general truth of the Collatz conjecture for* m^*.

Proof. (a) Let $s_i \equiv 3 \pmod 4$. Then $k(s_i, [3]) = 1$, so $s_{i+1} = (3s_i + 1)/2$. Consider $s_{i+1} + 1 = (3s_i+1)/2+1 = (3s_i+3)/2 = 3(s_i+1)/2$. Thus, $\mathrm{val}_2(s_{i+1}+1) = \mathrm{val}_2(3(s_i+1)/2) = \mathrm{val}_2(s_i+1)-1$. Since $s_0 \equiv 3 \pmod 4$, $s_0 + 1 \equiv 0 \pmod 4$, so $\mathrm{val}_2(s_0 + 1) \geq 2$. If the sequence $s_0, s_1, \ldots, s_{j_0-1}$ are all $\equiv 3 \pmod 4$, then $\mathrm{val}_2(s_{j_0} + 1) = \mathrm{val}_2(s_0 + 1) - j_0$. Since $s_{j_0} \in \mathbb{Z}^+_{\mathrm{odd}}$, $s_{j_0} + 1$ is a positive even integer, so $\mathrm{val}_2(s_{j_0} + 1) \geq 1$. The sequence must reach a state s_{j_0} such that $\mathrm{val}_2(s_{j_0} + 1) = 1$. If $\mathrm{val}_2(s_{j_0} + 1) = 1$, then $s_{j_0} + 1 \equiv 2 \pmod 4$, which implies $s_{j_0} \equiv 1 \pmod 4$. Thus, $\sigma_{j_0} = [1]$. This occurs when $j_0 = \mathrm{val}_2(s_0 + 1) - 1$. The number of steps j_0 is therefore finite and bounded.

(b) Let $s^* = s_{j_0}$ be the first term in the sequence such that $s^* \equiv 1 \pmod 4$. The subsequent step involves $m^* = (s^* - 1)/4$. We examine if $m^* < s_0$. The term s_{j_0} can be expressed as $s_{j_0} = \frac{3^{j_0}s_0+(3^{j_0}-2^{j_0})}{2^{j_0}}$. Then $m^* = \frac{s^*-1}{4} = \frac{1}{4}\left(\frac{3^{j_0}s_0+3^{j_0}-2^{j_0}}{2^{j_0}} - 1\right) = \frac{3^{j_0}s_0+3^{j_0}-2^{j_0+1}}{2^{j_0+2}}$. The condition $m^* < s_0$ is $3^{j_0}s_0 + 3^{j_0} - 2^{j_0+1} < s_0 \cdot 2^{j_0+2}$, which simplifies to $s_0(2^{j_0+2} - 3^{j_0}) > 3^{j_0} - 2^{j_0+1}$. This inequality holds if $2^{j_0+2} - 3^{j_0} > 0$ (i.e., $j_0 \leq 3$) and s_0 is not excessively small.

- If $j_0 = 1$ ($s_0 \equiv 3 \pmod 8$): $m^* = (3s_0 - 1)/8$. $m^* < s_0 \iff 5s_0 > -1$, true for $s_0 \in \mathbb{Z}^+_{\mathrm{odd}}$.
- If $j_0 = 2$ ($s_0 \equiv 7 \pmod{16}$ pattern): $m^* = (9s_0 + 1)/16$. $m^* < s_0 \iff 7s_0 > 1$, true for $s_0 \in \mathbb{Z}^+_{\mathrm{odd}}$.
- If $j_0 = 3$ ($s_0 \equiv 15 \pmod{32}$ pattern): $m^* = (27s_0 + 19)/32$. $m^* < s_0 \iff 5s_0 > 19$, true for $s_0 \geq 5$.
- If $j_0 = 4$ (e.g., $s_0 = 31 \equiv 31 \pmod{64}$ pattern): $m^* = (81s_0+55)/128$. For $s_0 = 31$, $s^* = 161$, $m^* = 40$. Here $m^* = 40 > s_0 = 31$.

Thus, $m^* < s_0$ is not universally guaranteed. When $m^* \geq s_0$, the inductive hypothesis cannot be directly applied to m^* to prove convergence for s_0 based on $m^* < s_0$. The convergence then relies on the Collatz conjecture holding true for m^*. □

Proof of Theorem 3.1. We proceed by strong induction on $s_0 \in \mathbb{Z}^+_{\text{odd}}$. Base Case: $s_0 = 1$. $T_X(1, [1]) = (1, [1])$. The sequence converges.

Inductive Hypothesis (IH): Assume that for all odd $j' \in \mathbb{Z}^+_{\text{odd}}$ with $1 \leq j' < s_0$, $T^k_X(j', [j' \pmod 4])$ reaches $(1, [1])$ for some k.

Consider $s_0 > 1$. Case 1: $s_0 \equiv 1 \pmod 4$. By Lemma 3.2, $T_X(s_0, [1]) = (s_1, \sigma_1)$ where $s_1 = T_C(m)$ and $m = (s_0 - 1)/4 < s_0$. If m is odd, by IH, the sequence from $(m, [m \pmod 4])$ converges. Since $s_1 = T_C(m)$ is the first term, the sequence from s_1 converges. If m is even ($m = 2^p j'_{odd}, p \geq 1$), then $T_C(m) = T_C(j'_{odd})$. Since $j'_{odd} \leq m < s_0$, by IH, the sequence from j'_{odd} converges. So the sequence from s_1 converges. Thus, if $s_0 \equiv 1 \pmod 4$, the sequence from (s_0, σ_0) converges.

Case 2: $s_0 \equiv 3 \pmod 4$. By Lemma 3.3(a), after $j_0 = \text{val}_2(s_0+1)-1$ steps, the sequence reaches $(s^*, [1])$ where $s^* \equiv 1 \pmod 4$. The next step is $T_X(s^*, [1]) = (s^{**}, \sigma^{**})$ where $s^{**} = T_C(m^*)$ and $m^* = (s^* - 1)/4$. As shown in Lemma 3.3(b), m^* is not always less than s_0. If $m^* < s_0$, then by IH (applied to m^* or its first odd part), the sequence from m^* converges, and thus the sequence from s_0 converges. If $m^* \geq s_0$ (e.g., for $s_0 = 31$, $m^* = 40$), the convergence of the sequence from s_0 depends on the convergence of the sequence from m^*. The Collatz conjecture asserts that all such sequences (including the one for m^*) converge to 1. Assuming the truth of the conjecture for m^*, the sequence for s_0 also converges.

This framework demonstrates that every Collatz sequence for $s_0 > 1$ either directly reduces its primary argument via $m = (s_0 - 1)/4 < s_0$, or it transitions in a finite, bounded number of steps to a state from which it reduces its primary argument to $m^* = (s^* - 1)/4$. The Collatz conjecture is true if and only if this process of generating m or m^* always leads to a sequence that eventually reaches 1. This is equivalent to the standard assertion that no Collatz sequence for $n \in \mathbb{Z}^+$ diverges or enters a cycle other than $1 \to 4 \to 2 \to 1$. □

4 Discussion and Conclusion

The state-augmented system $X = \mathbb{Z}^+_{\text{odd}} \times (\mathbb{Z}/4\mathbb{Z})$ provides a structured framework for analyzing the "shortcut" Collatz map T_C. The main contributions are:

1. A clear case distinction based on $s \pmod 4$:
   - If $s \equiv 1 \pmod 4$, the problem directly reduces to analyzing $T_C(m)$ for $m = (s - 1)/4$. Since $m < s$, this is a direct descent for the inductive argument.
   - If $s \equiv 3 \pmod 4$, the operator T_X applies $s \to (3s + 1)/2$ repeatedly. We proved this phase is finite, lasting $j_0 = \text{val}_2(s_0 + 1) - 1$ steps, until a state $(s^*, [1])$ with $s^* \equiv 1 \pmod 4$ is reached.
2. The problem then reduces to analyzing $T_C(m^*)$ for $m^* = (s^* - 1)/4$.

The Collatz conjecture hinges on the behavior of the sequence starting from m^*. While for many s_0, $m^* < s_0$ (allowing direct application of the inductive hypothesis), this is not universally true (e.g., $s_0 = 31 \implies m^* = 40$). In such cases, the convergence of the original sequence from s_0 relies on the (conjectured) convergence of the sequence from m^*.

This framework effectively reduces the Collatz conjecture to demonstrating that the "effective descent argument" m^* (or a term in its subsequent Collatz sequence) is always eventually smaller than the original s_0 that generated it through the $s \equiv 3 \pmod 4$ path. This is a known hard aspect of the conjecture, often referred to as proving "eventual decrease" or "non-divergence."

Computational examples illustrate this. For $s_0 = 31$: $j_0 = \mathrm{val}_2(31+1) - 1 = \mathrm{val}_2(32) - 1 = 5 - 1 = 4$. $s_0 = 31(\sigma = [3]) \xrightarrow{k=1} s_1 = 47(\sigma = [3]) \xrightarrow{k=1} s_2 = 71(\sigma = [3]) \xrightarrow{k=1} s_3 = 107(\sigma = [3]) \xrightarrow{k=1} s^* = s_4 = 161(\sigma = [1])$. Then $m^* = (161-1)/4 = 40$. Although $m^* = 40 > s_0 = 31$, the Collatz sequence for $m^* = 40$ is $40 \to 20 \to 10 \to 5 \to \cdots \to 1$. Since the sequence for m^* converges, the sequence for $s_0 = 31$ also converges.

This state-augmented analysis rigorously structures the initial steps of any Collatz sequence and pinpoints that the conjecture's truth rests on the global convergence behavior, specifically that no sequence can escape reduction to smaller values indefinitely. Prior analyses showing the unreachability of known 2-adic attractors outside $\mathbb{Z}^+$ from positive integers further support focusing on the dynamics within $\mathbb{Z}^+$.